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Proximinality in Banach spaces ☆

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Abstract

In this paper, we study approximatively τ-compact and τ-strongly Chebyshev sets, where τ is the norm or the weak topology. We show that the metric projection onto τ-strongly Chebyshev sets are norm-τ continuous. We characterize approximatively τ-compact and τ-strongly Chebyshev hyperplanes and use them to characterize factor reflexive proximinal subspaces in τ-almost locally uniformly rotund spaces. We also prove some stability results on approximatively τ-compact and τ-strongly Chebyshev subspaces.
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1. Introduction

Let X be a real Banach space. For a closed set K in X and $x \in X$, we denote the distance function of K at x by $d(x, K) = \inf\{\|x - k\|: k \in K\}$. The metric projection of x onto K is $P_K(x) = \{k \in K: \|x - k\| = d(x, K)\}$. The set K is called proximinal (respectively Chebyshev) if for every $x \in X \setminus K$, $P_K(x)$ is nonempty (respectively a singleton). It is known that a Banach space X is reflexive if and only if every closed convex subset of X is proximinal in X.

The notion of approximative compactness has been introduced by Efimov and Stechkin [6] (see also [4]). Deutsch [5] extended this notion to define approximatively τ-compact sets for a "regular mode of convergence τ" [5, Definition 2.3] where τ includes the norm, weak or weak* topology. However, there are many cases in which τ does not arise from any topology.

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In this paper, unless otherwise mentioned, we denote by τ either the norm or the weak topology on X. And we consider the following related notions of proximinality. Recall that a sequence $\{z_n\} \subseteq K$ is called minimizing for $x \in X \setminus K$, if $\|x - z_n\| \to d(x, K)$.

Definition 1.1. Let K be a τ-closed subset of X and $x_0 \in X \setminus K$.

(a) We say that K is approximatively τ-compact for x_0 if every minimizing sequence $\{z_n\} \subseteq K$ for x_0 has a τ-convergent subsequence.
(b) We say that K is τ-strongly Chebyshev for x_0 if every minimizing sequence $\{z_n\} \subseteq K$ for x_0 is τ-convergent.
If K is approximatively τ-compact (or τ-strongly Chebyshev) for every $x \in X \setminus K$, we say K is approximatively τ-compact (respectively τ-strongly Chebyshev) in X.
As usual, in case τ is the norm topology, we omit it.

We note that approximative τ-compactness implies proximinality. It is also clear that if K is τ-strongly Chebyshev for x_0, then $P_K(x_0)$ is a singleton.

Any compact set or any finite-dimensional subspace of a Banach space is clearly approximatively compact. Similarly, any weakly compact set or any reflexive subspace of a Banach space is clearly approximatively weakly compact.

The notion of strongly proximinal subspaces have been defined in [8]. We extend this to define the following:

Definition 1.2. For a closed set $K \subseteq X$, $x \in X$ and $\delta > 0$, let

$$P_K(x, \delta) = \{z \in K\colon \|z - x\| < d(x, K) + \delta\}.$$

A τ-closed set $K \subseteq X$ is τ-strongly proximinal for $x \in X \setminus K$, if K is proximinal for x and for any τ-neighbourhood V of 0 in X, there exists $\delta > 0$ such that $P_K(x, \delta) \subseteq P_K(x) + V$. If this holds for every $x \in X \setminus K$, then K is said to be τ-strongly proximinal in X.

Clearly, when K is a subspace and τ is the norm topology, we get back the definition of [8].

In Section 2, we study the relationship between approximatively τ-compact, τ-strongly proximinal and τ-strongly Chebyshev sets and prove that the metric projection onto τ-strongly Chebyshev sets are norm-τ continuous. We also characterize approximatively τ-compact and τ-strongly Chebyshev hyperplanes as kernels of τ-strongly support functionals (see Definition 2.9) and τ-strongly exposing functionals, respectively. We show that if Y is a reflexive Chebyshev subspace of X such that $P_Y^{-1}(0)$ is weakly closed, then P_Y is weak–weak continuous.

In Section 3, we consider proximinality in τ-almost locally uniformly rotund (τ-ALUR) Banach spaces. These spaces were defined in [2] (see Definition 3.1). From the results of [1], it follows that a Banach space X is τ-ALUR $\Leftrightarrow$ every norm-attaining functional $x^* \in X^*$ is a τ-strongly exposing functional. We refer to [1,2] for various characterizations and properties of such spaces.

It is well known that if a finite-codimensional subspace Y of X is proximinal in X, then $Y^\perp \subseteq NA(X)$, the set of all norm-attaining functionals in X^*. Conditions under which the converse holds have been the subject of many recent papers, e.g., [8,9,14]. We show that the converse holds in τ-ALUR spaces and, in fact, implies that Y is τ-strongly Chebyshev.

In Section 4, we consider some stability results on approximatively τ-compact and τ-strongly Chebyshev subspaces.

The closed unit ball and the unit sphere of X will be denoted by B_X and S_X, respectively. We will denote by $NA(X)$ the norm-attaining functionals in X^*. For a closed bounded convex set C, $\operatorname{ext} C$ denotes the set of extreme points of C. By a subspace, we mean a closed linear subspace.

2. General results

Note that if K is τ-closed, then K is τ-strongly Chebyshev $\Rightarrow$ K is approximatively τ-compact $\Rightarrow$ K is proximinal. And none of the implications can be reversed (see Example 2.1 and Theorem 2.8).

Indeed, if a minimizing sequence $\{z_n\}$ for x has a τ-convergent subsequence $\{z_{n_k}\}$ with τ-limit point $z \in K$, then, since the norm is τ-lower semicontinuous,

$$d(x, K) \leqslant \|x - z\| \leqslant \liminf \|x - z_{n_k}\| = d(x, K).$$

Thus, $z \in P_K(x)$.

Example 2.1. Let $X = \ell_\infty$, $Y = c_0$. Then Y, being an M-ideal in X, is proximinal in X. For $x_0 = (1, 1, 1, \dots) \in \ell_\infty$, the sequence $y_n = (1, 1, \dots, 1, 0, 0, \dots) \in Y$ is minimizing, but $\{y_n\}$ has no τ-convergent subsequence. Thus proximinality does not imply approximative τ-compactness.

As suggested by the name, τ-strongly Chebyshev sets are precisely sets that are τ-strongly proximinal and Chebyshev. Indeed, we have

Theorem 2.2. *Let K be a τ-closed subset of Banach space X and $x_0 \in X \setminus K$. Then K is approximatively τ-compact for $x_0 \Leftrightarrow K$ is τ-strongly proximinal for x_0 and $P_K(x_0)$ is τ-compact.*

Proof. Notice that in τ-topology, compactness and sequential compactness coincide. It follows that if K is approximatively τ-compact for x_0, then $P_K(x_0)$ is τ-compact.

If K is not τ-strongly proximinal for x_0, then there exist a τ-neighbourhood V of 0 and a minimizing sequence $\{z_n\} \subseteq K$ with $z_n \notin P_K(x_0) + V$ for all $n \geqslant 1$. Since K is approximatively τ-compact for x_0, there is a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z_0 \in K$ in τ. Then $z_0 \in P_K(x_0)$ and so, $z_n \in z_0 + V \subseteq P_K(x_0) + V$ for some $n \geqslant 1$. A contradiction!

Conversely, suppose K is τ-strongly proximinal for x_0 and $P_K(x_0)$ is τ-compact, but K is not approximatively τ-compact for x_0. Then there is a minimizing sequence $\{z_n\} \subseteq K$ such that no subsequence is τ-convergent. It follows that for any $z \in P_K(x_0)$, there is a τ-neighbourhood U_z of z and $N_z \in \mathbb{N}$ such that for all $n \geqslant N_z$, $z_n \notin U_z$. Since $P_K(x_0)$ is τ-compact, there is a τ-neighbourhood V of 0 and $N_0 \in \mathbb{N}$ such that for all $n \geqslant N_0$, $z_n \notin P_K(x_0) + V$. Since K is τ-strongly proximinal for x_0, there exists $\delta > 0$ such that $P_K(x_0, \delta) \subseteq P_K(x_0) + V$. Note that for any minimizing sequence $\{z_n\} \subseteq K$, $z_n \in P_K(x, \delta)$ eventually. This is a contradiction! □

Theorem 2.3. *Let K be a τ-closed subset of Banach space X and $x_0 \in X \setminus K$. Then the following are equivalent*:

(a) *K is τ-strongly Chebyshev for x_0.*
(b) *K is τ-strongly proximinal for x_0 and $P_K(x_0)$ is a singleton.*
(c) *K is approximatively τ-compact for x_0 and $P_K(x_0)$ is a singleton.*

Proof. By Theorem 2.2, it suffices to show (b) $\Rightarrow$ (a).

Suppose K is τ-strongly proximinal for x_0 and $P_K(x_0) = \{z_0\}$. Let V be a τ-neighbourhood of 0. Since K is τ-strongly proximinal for x_0, there exists $\delta > 0$ such that $P_K(x_0, \delta) \subseteq z_0 + V$. Thus, for any minimizing sequence $\{z_n\} \subseteq K$, $z_n \in P_K(x, \delta) \subseteq z_0 + V$ for sufficiently large n. Hence $z_n \to z_0$ in τ. □

Example 2.4. It is shown in [11, Proposition IV.1.14] that if Y is a proximinal subspace of X such that for every $x \in X$, $P_Y(x)$ is weakly compact, then

$$\operatorname{card}(\operatorname{ext} B_{X/Y}) \leqslant \operatorname{card}(\operatorname{ext} B_X).$$

For $X = c_0$ or $L_1(\mu)$ with μ nonatomic, since $\operatorname{ext} B_X = \emptyset$, it follows that X has no Chebyshev or approximatively τ-compact subspace of finite-codimension. Example 2.1 is a special case of this general phenomenon. On the other hand, it is known that any proximinal subspace of finite-codimension in c_0 is strongly proximinal [8, Theorem 3.4]. Thus, strong proximinality does not imply approximative τ-compactness.

And, for $X = C_{\mathbb{R}}([0, 1])$, since $\operatorname{card}(\operatorname{ext} B_X) = 2$, it follows that X has no Chebyshev or approximatively τ-compact subspace of finite-codimension $\geqslant 2$ (also see [15, pp. 321–324]).

Regarding continuity of the metric projection, we have the following result.

Theorem 2.5. *Let K be a τ-closed subset of a Banach space X and $x_0 \in X \setminus K$.*

(a) *K is τ-strongly Chebyshev for $x_0 \Rightarrow$ the metric projection P_K is single-valued and norm-τ continuous at x_0, i.e., if $x_n \to x_0$ in norm and $z_n \in P_K(x_n)$, then $z_n \to z_0 = P_K(x_0)$ in τ.*
(b) *K is approximatively τ-compact for $x_0 \Rightarrow$ the metric projection P_K is norm-τ upper semicontinuous (usc) at x_0, i.e., for any τ-open set W with $P_K(x_0) \subseteq W$, there exists $\varepsilon > 0$ such that $P_K(x) \subseteq W$ whenever $\|x - x_0\| < \varepsilon$.*

Proof. (a) If K is τ-strongly Chebyshev for x_0, then $P_K(x_0)$ is a singleton, say $\{z_0\}$. Observe that, if $x_n \to x_0$ and $z_n \in P_K(x_n)$, then $\{z_n\}$ is a minimizing sequence for x_0. Therefore, $z_n \to z_0$ in τ-topology.

(b) Assume K is approximatively τ-compact for x_0. Then by Theorem 2.2, K is τ-strongly proximinal for x_0 and $P_K(x_0)$ is τ-compact.

Let W be a τ-open set such that $P_K(x_0) \subseteq W$. Since $P_K(x_0)$ is τ-compact, there exists a τ-open neighbourhood V of 0 such that $P_K(x_0) + V \subseteq W$. Since K is τ-strongly proximinal for x_0, there exists $\delta > 0$ such that $P_K(x_0, \delta) \subseteq P_K(x_0) + V \subseteq W$. Clearly, for $\|x - x_0\| < \delta/2$, $P_K(x) \subseteq P_K(x_0, \delta) \subseteq W$. □

Remark 2.6. Let τ be a topology on X such that B_X is τ-closed. This corresponds to a "regular mode of convergence" that is "topological" [5]. Examples include, apart from the norm, weak and weak* topologies, the strong and weak operator topologies on the space $\mathcal{L}(X, Y)$. If we define approximatively τ-compact sets and τ-strongly Chebyshev sets in terms of minimizing nets rather than sequences, the modified definition for τ-strongly Chebyshev sets is easily seen to be equivalent to the original one. And all the above results still hold. If a "regular mode of convergence" is not "topological," we do not know if Theorems 2.2 and 2.5(b) will still hold.

Similarly, if we call the sequential version, approximative τ-sequential compactness, then a τ-closed set K is approximatively τ-sequentially compact for $x_0^* \Rightarrow K$ is τ-strongly proximinal for x_0^* and $P_K(x_0^*)$ is τ-sequentially compact. We do not know if the converse hold in general. Still, a τ-closed set K is approximatively τ-sequentially compact for $x_0^* \Rightarrow$ the metric projection P_K is norm-τ usc for x_0^*. This follows from [5, Theorem 2.7] and gives an alternative proof of Theorem 2.5(b).

When τ is the norm topology, we have

Proposition 2.7. *Let X be a Banach space, K be a closed subset of X and $x_0 \in X \setminus K$. Then K is strongly Chebyshev for $x_0 \Leftrightarrow$ for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\cdot\|$-diam$(P_K(x_0, \delta)) < \varepsilon$.*

Proof. If K is strongly Chebyshev for x_0, then $P_K(x_0) = \{z_0\}$. Therefore every minimizing sequence converges to z_0. Now, if the given condition does not hold, there exists $\varepsilon > 0$ and $z_n \in P_K(x_0, 1/n)$ such that $\|z_n - z_0\| \geqslant \varepsilon$. Then $\{z_n\}$ is a minimizing sequence that does not converge to z_0. Contradiction!

Conversely, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\cdot\|$-diam$(P_K(x_0, \delta)) < \varepsilon$, then the Cantor Intersection Theorem shows that $P_K(x_0)$ is a singleton $\{z_0\}$ (say). Suppose $\{z_n\} \subseteq K$ is a minimizing sequence for x_0. Let $\varepsilon > 0$ and let $\delta > 0$ be such that $\|\cdot\|$-diam$(P_K(x_0, \delta)) < \varepsilon$. Then, for all sufficiently large n, $z_n \in P_K(x_0, \delta)$. Therefore, $\|z_n - z_0\| \leqslant \|\cdot\|$-diam$(P_K(x_0, \delta)) < \varepsilon$. □

Fan and Glicksberg [7] showed that in a strictly convex reflexive space X with the Kadec–Klee (KK) property (i.e., $x_n, x \in B_X$, $\lim_n \|x_n\| = \|x\| = 1$ and w-$\lim_n x_n = x \Rightarrow \lim_n \|x_n - x\| = 0$), the metric projection P_K is norm–norm continuous for all nonempty closed convex sets $K \subseteq X$ and they have given various characterizations for such spaces. The following result characterizes reflexive spaces from proximinality point of view. The proof follows from arguments in [5,7].

Theorem 2.8. *Let X be a Banach space.*

(a) *Every closed convex set is proximinal in $X \Leftrightarrow X$ is reflexive $\Leftrightarrow$ every closed convex set is approximatively weakly compact in X.*
(b) *Every closed convex set is weakly strongly Chebyshev in $X \Leftrightarrow X$ is reflexive and strictly convex.*
(c) *Every closed convex set is approximatively compact in $X \Leftrightarrow X$ is reflexive and has the KK property.*
(d) *Every closed convex set is strongly Chebyshev in $X \Leftrightarrow X$ is reflexive, strictly convex and has the KK property.*

A.L. Brown [3] has shown that there exists a separable strictly convex and reflexive real Banach space X and a closed subspace Y such that P_Y is not norm–norm continuous.

We now characterize approximatively τ-compact and τ-strongly Chebyshev hyperplanes.

Definition 2.9.

(a) Recall that $x^* \in S_{X^*}$ is called a τ-strongly exposing functional if $x^* \in NA(X)$ and every $\{x_n\} \subseteq B_X$ with $\lim x^*(x_n) = 1$ is τ-convergent.
(b) Let us call $x^* \in S_{X^*}$ a τ-strongly support functional if $x^* \in NA(X)$ and every $\{x_n\} \subseteq B_X$ with $\lim x^*(x_n) = 1$ has a τ-convergent subsequence.

Theorem 2.10. *For a Banach space X and $x^* \in S_{X^*}$, $\ker x^*$ is approximatively τ-compact* (*respectively τ-strongly Chebyshev*) *in $X \Leftrightarrow x^*$ is a τ-strongly support* (*respectively τ-strongly exposing*) *functional.*

Proof. Suppose $Y = \ker x^*$ is approximatively τ-compact. Then Y is proximinal, and hence, $x^* \in NA(X)$. Thus, there exists $x_0 \in S_X$ such that $x^*(x_0) = 1$. Let $\{x_n\} \subseteq S_X$ such that $x^*(x_n) \to 1$. Let $y_n = x^*(x_n)x_0 - x_n \in Y$. Then

$$1 = x^*(x_0) = x^*(x_0 - y_n) \leqslant \|x_0 - y_n\| = \big\|x_0 - x^*(x_n)x_0 + x_n\big\| \leqslant \big(1 - x^*(x_n)\big) + 1 \to 1.$$

Thus, $\{y_n\}$ is a minimizing sequence for x_0, and hence has a τ-convergent subsequence. It follows that so does $\{x_n\}$.

Conversely, suppose x^* is a τ-strongly support functional. Then, $x^* \in NA(X)$, and hence, $Y = \ker x^*$ is a proximinal subspace. Indeed, let $x_0 \in S_X$ be such that $x^*(x_0) = 1$. Then for any $x \in X$, $d(x, Y) = |x^*(x)|$ and $x - x^*(x)x_0 \in P_Y(x)$.

Let $\{z_n\} \subseteq Y$ be a minimizing sequence for $x \in X \setminus Y$. It follows that $\|x - z_n\| \to d(x, Y) = |x^*(x)| \neq 0$ and $x^*(x - z_n) = x^*(x)$. Let $w_n = \eta_x(x - z_n)/\|x - z_n\|$, where η_x is the sign of $x^*(x)$, i.e., $\eta_x x^*(x) = |x^*(x)|$. Then $\|w_n\| = 1$ and the sequence $x^*(w_n) \to 1$. Since x^* is a τ-strongly support functional, the sequence $\{w_n\}$ has a τ-convergent subsequence. And hence, $\{z_n\}$ has a τ-convergent subsequence too.

The other case is similar. □

The following is the analogue of Theorem 2.8 without reflexivity.

Theorem 2.11. *For a Banach space X, the following are equivalent:*

(a) *Every $x^* \in NA(X)$ is a τ-strongly support* (*respectively τ-strongly exposing*) *functional.*
(b) *Every proximinal convex set $K \subseteq X$ is approximatively τ-compact* (*respectively τ-strongly Chebyshev*).
(c) *Every proximinal subspace $Y \subseteq X$ is approximatively τ-compact* (*respectively τ-strongly Chebyshev*).
(d) *Every proximinal hyperplane is approximatively τ-compact* (*respectively τ-strongly Chebyshev*).

Proof. (a) $\Rightarrow$ (b). Suppose every $x^* \in NA(X)$ is a τ-strongly support functional. Let K be a proximinal convex set in X and $x_0 \in X \setminus K$. Without loss of generality, we may assume $x_0 = 0$ and $d(0, K) = 1$. Then there exists $z_0 \in K$ such that $\|z_0\| = 1 = d(0, K)$. Since $d(0, K) = 1$, K and the open unit ball of X are disjoint convex sets and therefore, there exists $x^* \in S_{X^*}$ such that $\inf x^*(K) \geqslant 1$. It follows that $x^*(z_0) = 1$, that is, $x^* \in NA(X)$. Thus, x^* is a τ-strongly support functional. Let $\{z_n\} \subseteq K$ be a minimizing sequence for 0. Then $1 \leqslant x^*(z_n) \leqslant \|z_n\| \to 1$. Therefore, $\{z_n/\|z_n\|\}$, and hence $\{z_n\}$, has a τ-convergent subsequence.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) is trivial, and (d) $\Rightarrow$ (a) follows from Theorem 2.10. □

Remark 2.12. Observe that a subspace Y need not be strongly Chebyshev even if Y is a Chebyshev subspace of X and P_Y is norm–norm continuous. For example, let $x^* \in S_{X^*}$ be an exposing functional that is not strongly exposing. Then x^* attains its norm at a *unique* point $x_0 \in S_X$. Thus, $Y = \ker x^*$ is Chebyshev with $P_Y(x) = x - x^*(x)x_0$ for any $x \in X$, so that P_Y is continuous. But since x^* is not strongly exposing, by Theorem 2.10, Y is not strongly Chebyshev.

We do not know if Theorem 2.11 has an analogue for a "regular mode of convergence," "topological" or otherwise.

Definition 2.13. For $x^* \in X^* \setminus \{0\}$ and $x \in X$, let us define the following maps:

$$D(x) = \{x^* \in S_{X^*}\colon\ x^*(x) = \|x\|\},$$
$$D^{-1}(x^*) = \{x \in S_X\colon\ x^*(x) = \|x^*\|\}.$$

D is called the duality map and D^{-1} is called the pre-duality map. Naturally, D^{-1} is defined only on $NA(X)$.

Theorem 2.14. *Let X be a Banach space and $x_0^* \in S_{X^*}$.*

(a) *x_0^* is a τ-strongly exposing functional $\Rightarrow$ the pre-duality map D^{-1} is single-valued and norm-τ continuous at x_0^*, i.e., if $x_n^* \to x_0^*$ in norm and $x_n \in D^{-1}(x_n^*)$, then $x_n \to x_0 = D^{-1}(x_0^*)$ in τ.*
(b) *x_0^* is a τ-strongly support functional $\Rightarrow$ the pre-duality map D^{-1} is norm-τ usc at x_0^*, i.e., for any τ-open set W with $D^{-1}(x_0^*) \subseteq W$, there exists $\varepsilon > 0$ such that $D^{-1}(x^*) \subseteq W$ whenever $\|x^* - x_0^*\| < \varepsilon$.*

Proof. Observe that if $x_0^* \in NA(X)$ and $Y = \ker x_0^*$, then Y is proximinal with

$$x - P_Y(x) = x_0^*(x) D^{-1}(x_0^*) \quad \text{for any } x \in X.$$

Therefore, $P_Y(x)$ is a singleton $\Leftrightarrow D^{-1}(x_0^*)$ is a singleton. Thus (b) $\Rightarrow$ (a).

(b) Let x_0^* be a τ-strongly support functional. Then $\ker x_0^* = Y$ is approximatively τ-compact in X. Suppose that D^{-1} is not norm-τ usc at x_0^*. Then for some τ-open set W in X with $D^{-1}(x_0^*) \subseteq W$, there exists $\{x_n^*\} \subseteq S_{X^*}$ such that $\|x_n^* - x_0^*\| \to 0$ and $D^{-1}(x_n^*) \nsubseteq W$ for all n.

Let $z_n \in D^{-1}(x_n^*) \setminus W$. Fix $x_0 \in D^{-1}(x_0^*)$. Let $x_n = x_0^*(z_n)x_0 - z_n$. Then $\{x_n\} \subseteq Y$ is a minimizing sequence for x_0. Since Y approximatively τ-compact, $\{x_n\}$ has a τ-convergent subsequence. So $\{z_n\}$ has a τ-convergent subsequence converging to some z_0. Thus $z_0 \in D^{-1}(x_0^*) \subseteq W$, but $z_n \in X \setminus W$ and $X \setminus W$ is τ-closed, a contradiction. $\square$

Remark 2.15. It is well known that strongly exposing functionals are precisely the points of S_{X^*} at which the dual norm is Fréchet differentiable. Is there an analogous characterization of strongly support functionals?

Notice that one of the natural candidates, namely, the points of S_{X^*} at which the dual norm is strongly subdifferentiable (SSD) is actually weaker. To see this, recall that the dual norm is SSD at every $x^* \in NA(X) \Leftrightarrow$ every proximinal hyperplane is strongly proximinal [8, Proposition 2.6] and that this condition holds in c_0. Compare this with Theorem 2.11 in the light of Example 2.4.

We conclude this section with a result about when P_Y is weak–weak continuous. In [13], the authors prove that if Y is a finite-dimensional Chebyshev subspace of X such that $P_Y^{-1}(0)$ is weakly closed, then P_Y is weak–weak continuous. The following theorem improves this result.

Theorem 2.16. *If Y is a reflexive Chebyshev subspace of X such that $P_Y^{-1}(0)$ is weakly closed, then P_Y is weak–weak continuous.*

Proof. For any $x \in X$, we have $\|x - P_Y(x) - 0\| = d(x, Y) = d(x - P_Y(x), Y)$, so $x - P_Y(x) \in P_Y^{-1}(0)$. Now if $\{x_\alpha\} \subseteq X$ and $x_\alpha \to x$ weakly, then $\{x_\alpha\}$ is norm-bounded, i.e., $\|x_\alpha\| \leqslant C$ for some $C > 0$. Since $\|x_\alpha - P_Y(x_\alpha)\| = d(x_\alpha, Y) \leqslant \|x_\alpha\| \leqslant C$, $\|P_Y(x_\alpha)\| \leqslant 2C$. Since Y is reflexive, we have a subnet $P_Y(x_\gamma)$ that converges weakly to some $z \in Y$. Then $x_\gamma - P_Y(x_\gamma) \to x - z$ weakly. Since $P_Y^{-1}(0)$ is weakly closed and $x_\alpha - P_Y(x_\alpha) \in P_Y^{-1}(0)$, we have $x - z \in P_Y^{-1}(0)$, so that $\|x - z\| = d(x - z, Y) = d(x, Y)$. Therefore, $z \in P_Y(x)$. Since Y is a Chebyshev subspace of X, $P_Y(x)$ is a singleton. Thus, all weakly convergent subnets of $P_Y(x_\alpha)$ weakly converge to z. It follows that $P_Y(x_\alpha)$ weakly converges to $P_Y(x)$. $\square$

3. Proximinality in τ-ALUR Banach spaces

Definition 3.1. (See [2].) Recall that a Banach space X is τ-almost locally uniformly rotund (τ-ALUR) if for any $x \in S_X$, $\{x_n\} \subseteq B_X$ and $\{x_m^*\} \subseteq B_{X^*}$, the condition

$$\lim_m \lim_n x_m^*\left(\frac{x_n + x}{2}\right) = 1$$

implies τ-$\lim_n x_n = x$.

In the literature, the acronym ALUR is also used for average locally uniformly rotund which is known to be equivalent to X is strictly convex and has the KK property. We will use it in the above sense only. Theorem 3.4 below discusses the relation between these two notions.

In this section, we consider proximinality in τ-ALUR spaces.

From the results of [1, Proposition 4.4 and Corollary 4.6], it follows that a Banach space X is τ-ALUR $\Leftrightarrow$ every $x^* \in NA(X)$ is a τ-strongly exposing functional. Thus, Theorem 2.11 gives a new characterization of τ-ALUR spaces.

As noted earlier, if a finite-codimensional subspace Y of X is proximinal in X, then $Y^\perp \subseteq NA(X)$. We now show that the converse holds in τ-ALUR spaces. It is not difficult to show that the condition $Y^\perp \subseteq NA(X)$ forces X/Y to be reflexive [14, Lemma 2.2]. Thus, our result improves [9, Corollary 3.4].

Theorem 3.2. *Let X be a τ-ALUR Banach space and Y be a subspace such that X/Y is reflexive. Then the following are equivalent*:

(a) *Y is proximinal in X*;
(b) *Y is τ-strongly proximinal in X*;
(c) *Y is τ-strongly Chebyshev in X*;
(d) *$Y^\perp \subseteq NA(X)$.*

Proof. By Theorem 2.11, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (d). Since X/Y is reflexive, every $x^* \in Y^\perp \simeq (X/Y)^*$ is norm attaining on X/Y. Since Y is proximinal in X, $x^* \in NA(X)$. Thus $Y^\perp \subseteq NA(X)$.

(d) $\Rightarrow$ (a). Let $Y^\perp \subseteq NA(X)$. Since X is τ-ALUR, every $x^* \in Y^\perp \simeq (X/Y)^*$ is in particular an exposing functional. Since X/Y is reflexive and $X/Y = (Y^\perp)^*$, it is strictly convex. Let $Q : X \to X/Y$ be the quotient map. If $t \in S_{X/Y}$, there is $x^* \in Y^\perp$ with $\|x^*\| = x^*(t) = 1$ and since X/Y is strictly convex, $\{u \in S_{X/Y}\colon x^*(u) = \|u\| = 1\} = \{t\}$. Now since $x^* \in Y^\perp \subseteq NA(X)$, there is $x \in X$ with $\|x\| = x^*(x) = 1$, and thus $Q(x) = t$. Hence $Q(B_X) = B_{X/Y}$ and Y is proximinal in X. $\square$

Remark 3.3. The same proof shows that if Y is a subspace of X such that any $x^* \in Y^\perp$ is a τ-strongly exposing functional, then Y is τ-strongly Chebyshev.

Let Y be a finite-codimensional proximinal subspace of X. Suppose $Y^\perp$ has a basis consisting of τ-strongly exposing functionals. Does it imply that Y is τ-strongly Chebyshev in X?

We now compare ALUR with some related convexity notions.

Theorem 3.4. *For a Banach space X, X is LUR $\Rightarrow$ X is ALUR $\Rightarrow$ X is strictly convex and has the KK property. And neither converse implication holds.*

Proof. The fact that LUR $\Rightarrow$ ALUR, but not conversely, has been noted in [2]. Indeed, LUR $\Rightarrow$ ALUR is clear from the definitions. And as noted in [2], if X is an infinite-dimensional Banach space with separable dual, then there exists an equivalent norm on X such that X is ALUR but fails to be LUR.

Assume now that X is ALUR. Then X is clearly strictly convex. Let $\{x_n\}, x_0 \subseteq S_X$ be such that $x_n \to x_0$ weakly. Let $x^* \in S_{X^*}$ strongly expose x_0. Then $x^*(x_n) \to x^*(x_0) = 1$ so that $x_n \to x_0$ in norm. Hence X has the KK property.

Let $X = c_0$ and let $T : \ell_2 \to X$ be a weakly compact operator with dense range. Define

$$|||x^*||| = \|x^*\|_1 + \|T^*x^*\|_2.$$

By [11, Proposition III.2.11], this defines an equivalent dual norm on X^* such that c_0 is an M-ideal in ℓ_∞. This new norm on X^* is strictly convex and has the KK property. Now we show that there are points in $NA(X^*)$ that are not even w*-exposing for $B_{X^{***}}$.

Indeed X, with the $|||\cdot|||$-norm, is an M-ideal in X^{**}. So X^{***} can be decomposed as $X^{***} = X^* \oplus_1 X^\perp$. By Bishop–Phelps Theorem, there is $x^{***} \in S_{X^{***}} \setminus (X^* \cup X^\perp)$ such that $x^{***} \in NA(X^{**})$. Then there is $x^{**} \in X^{**}$ such that $x^{***}(x^{**}) = |||x^{**}||| = 1$. Now, we can write $x^{***} = x^* + z^{***}$ with $x^* \neq 0 \in X^*$, $z^{***} \neq 0 \in X^\perp$ and $1 = |||x^{***}||| = |||x^*||| + |||z^{***}|||$. It follows that $x^{**}(x^*) = |||x^*|||$, so that $x^{**} \in NA(X^*)$, but x^{**} does not w*-expose $x^*/|||x^*|||$ in $B_{X^{***}}$. □

Remark 3.5. Note that the last part of the proof shows that if X is not reflexive and an M-ideal in X^{**}, then X^* always fails to be wALUR.

We conclude this section with the following result.

Theorem 3.6. *An ALUR Banach space X is reflexive $\Leftrightarrow$ the intersection of any two proximinal hyperplanes is proximinal.*

Proof. If X is reflexive, then all subspaces are proximinal.

Conversely, let X be ALUR and suppose the intersection of any two proximinal hyperplanes is proximinal. Let $x^*, y^* \in NA(X)$. Then $\ker x^*$ and $\ker y^*$ are proximinal hyperplanes in X. If $Y = \ker x^* \cap \ker y^*$ is proximinal in X, then $Y^\perp \subseteq NA(X)$, and therefore, $\alpha x^* + \beta y^* \in NA(X)$ for all $\alpha, \beta \in \mathbb{R}$. This implies $NA(X)$ is a linear subspace of X^*. Since X is ALUR and since the set of strongly exposing functionals form a G_δ set, $NA(X)$ is a dense G_δ in X^*. Thus, by the Baire Category Theorem, for every $x^* \in X^*$, we have $(x^* + NA(X)) \cap NA(X) \neq \emptyset$. Hence $NA(X) - NA(X) = X^*$. Since $NA(X)$ is a linear subspace of X^*, $NA(X) = X^*$, which, by James' theorem, implies that X is reflexive. □

4. Stability results

In this section, we prove some stability results on approximatively τ-compact (respectively τ-strongly Chebyshev) subspaces.

Theorem 4.1. *Let $\{X_i\colon 1 \leqslant i \leqslant m\}$ be a family of Banach spaces and Y_i be a subspace of X_i, respectively, for $1 \leqslant i \leqslant m$. Consider $X = \bigoplus_{\ell_p} X_i$ and $Y = \bigoplus_{\ell_p} Y_i$, $1 \leqslant p < \infty$. Then Y is approximatively τ-compact (respectively τ-strongly Chebyshev) in X $\Leftrightarrow$ Y_i is approximatively τ-compact (respectively τ-strongly Chebyshev) in X_i for all $1 \leqslant i \leqslant m$.*

Proof. We will prove the result for strongly Chebyshev subspaces. It will be clear that the remaining cases are similar.

Suppose Y is strongly Chebyshev in X. Fix $1 \leqslant i \leqslant m$. Let $x_i \in X_i$ such that $d(x_i, Y_i) = 1$. Let $x = (0, \dots, 0, x_i, 0, \dots, 0)$. Then for any $y \in Y$, $\|x - y\|_p^p = \sum_{j \neq i} \|y_j\|^p + \|x_i - y_i\|^p \geqslant \|x_i - y_i\|^p$. It follows that $d(x, Y) = 1$ and the nearest point y_0 must be of the form $y_0 = (0, \dots, 0, y_i, 0, \dots, 0)$ with y_i nearest to x_i in Y_i. Thus, Y_i is proximinal in X_i.

To prove that Y_i is strongly Chebyshev in X_i, let $\{y_{n,i}\}$ be a minimizing sequence for x_i. Let $y_n = (0, \dots, 0, y_{n,i}, 0, \dots, 0)$. Then $\{y_n\}$ is a minimizing sequence for x. So $\{y_n\}$ converges in Y and this implies that $\{y_{n,i}\}$ converges in Y_i proving that Y_i is strongly Chebyshev in X_i for $1 \leqslant i \leqslant m$.

Conversely, suppose for all $1 \leqslant i \leqslant n$, Y_i is strongly Chebyshev in X_i. First we prove that Y is proximinal in X. Let $x = (x_i)_{1 \leqslant i \leqslant n} \in X$. For every $1 \leqslant i \leqslant m$, there exists $y_i \in Y_i$ such that $\|x_i - y_i\| = d(x_i, Y_i)$. Let $y = (y_i)_{1 \leqslant i \leqslant n} \in Y$. Then for any $z = (z_i)_{1 \leqslant i \leqslant n} \in Y$, $\|x - z\|_p^p = \sum_{i=1}^n \|x_i - z_i\|^p \geqslant \sum_{i=1}^m \|x_i - y_i\|^p = \|x - y\|_p^p$. Thus, y is nearest to x.

Now we claim that Y is strongly Chebyshev. Let $x = (x_i)_{1 \leqslant i \leqslant m} \in X$ such that $d(x, Y) = 1$ and $\{y_n = \{y_{n,i}\}\}$ be a minimizing sequence for x. Clearly $\{y_{n,i}\}$ is minimizing sequence for Y_i for $1 \leqslant i \leqslant m$. Hence it converges in Y_i for $1 \leqslant i \leqslant m$. Now it is easy to see that $\{y_n\}$ is a converging minimizing sequence for x. □

Corollary 4.2. *Let Λ be an index set. For all $\lambda \in \Lambda$, let Y_λ be a subspace of a Banach space X_λ. For $1 \leqslant p < \infty$, let $X = \bigoplus_{\ell_p} X_\lambda$ and $Y = \bigoplus_{\ell_p} Y_\lambda$.*

(a) *Y is approximatively τ-compact (respectively τ-strongly Chebyshev) in $X \Rightarrow Y_\lambda$ is approximatively τ-compact (respectively τ-strongly Chebyshev) in X_λ for all $\lambda \in \Lambda$.*
(b) *Y is proximinal in $X \Leftrightarrow Y_\lambda$ is proximinal in X_λ for all $\lambda \in \Lambda$.*
(c) *Y is Chebyshev in $X \Leftrightarrow Y_\lambda$ is Chebyshev in X_λ for all $\lambda \in \Lambda$.*

Proof. The proof of (a) and the "necessity" part of (b) and (c) is essentially contained in the proof of Theorem 4.1.

For "sufficiency" in both (b) and (c), let $x = (x_\lambda) \in X$. For every $\lambda \in \Lambda$, there exists $y_\lambda \in Y_\lambda$ such that $\|x_\lambda - y_\lambda\| = d(x_\lambda, Y_\lambda)$. Let $y = (y_\lambda)$. It actually suffices to note that $y \in Y$. This follows from the simple observation that $\|x_\lambda - y_\lambda\| = d(x_\lambda, Y_\lambda) \leqslant \|x_\lambda\|$ and $x = (x_\lambda) \in X$. □

Remark 4.3. The above theorem and corollary do not hold for $p = \infty$. Indeed if $X_i = \mathbb{R}$, $i = 1, 2, 3$, and $Y_1 = Y_2 = \mathbb{R}$, $Y_3 = \{0\}$. Let X be the ℓ_∞ sum of X_i, $i = 1, 2, 3$, and Y be the ℓ_∞ sum of Y_i. Then Y_i is strongly Chebyshev in X_i for $i = 1, 2, 3$, but Y is not strongly Chebyshev as it is not Chebyshev subspace in X.

Remark 4.4. Let X be a Banach space and let μ be a Lebesgue measure on $[0, 1]$. In [10,12], it is proved that f is a strongly exposed point of the unit ball of Lebesgue–Bochner function space $L^p(\mu, X)$, $1 < p < \infty$, if and only if f is a unit vector and $f(t)/\|f(t)\|$ is strongly exposed point of the unit ball of X for almost all t in the support of f. It follows that X is ALUR if and only if $L^p(\mu, X)$ is ALUR. And X admits strongly Chebyshev hyperplanes if and only if $L^p(\mu, X)$ does.

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